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Alikhanov Legendre—Galerkin Spectral Method for the Coupled Nonlinear Time-Space Fractional Ginzburg–Landau Complex System

Mahmoud A. Zaky [1,*], Ahmed S. Hendy [2,3,*] and Rob H. De Staelen [4,5,*]

[1] Department of Applied Mathematics, National Research Centre, Dokki, Cairo 12622, Egypt
[2] Department of Computational Mathematics and Computer Science, Institute of Natural Sciences and Mathematics, Ural Federal University, 19 Mira St., 620002 Yekaterinburg, Russia
[3] Department of Mathematics, Faculty of Science Benha University, Benha 13511, Egypt
[4] Dean's Office of the Faculty of Medicine and Health Sciences, Ghent University, C. Heymanslaan 10, 9000 Gent, Belgium
[5] Beheer en Algemene Directie, Ghent University Hospital, C. Heymanslaan 10, 9000 Gent, Belgium
* Correspondence: ma.zaky@yahoo.com (M.A.Z.); ahmed.hendy@fsc.bu.edu.eg (A.S.H.); rob.destaelen@ugent.be (R.H.D.S.)

Abstract: A finite difference/Galerkin spectral discretization for the temporal and spatial fractional coupled Ginzburg–Landau system is proposed and analyzed. The Alikhanov $L2\text{-}1_\sigma$ difference formula is utilized to discretize the time Caputo fractional derivative, while the Legendre-Galerkin spectral approximation is used to approximate the Riesz spatial fractional operator. The scheme is shown efficiently applicable with spectral accuracy in space and second-order in time. A discrete form of the fractional Grönwall inequality is applied to establish the error estimates of the approximate solution based on the discrete energy estimates technique. The key aspects of the implementation of the numerical continuation are complemented with some numerical experiments to confirm the theoretical claims.

Keywords: generalized fractional coupled Ginzburg–Landau system; Alikhanov difference formula; Galerkin spectral scheme; discrete fractional Grönwall inequality

MSC: 65M06; 35K15; 35K55; 35K57

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Citation: Zaky, M.A.; Hendy, A.S.; De Staelen, R.H. Alikhanov Legendre–Galerkin Spectral Method for the Coupled Nonlinear Time-Space Fractional Ginzburg–Landau Complex System. *Mathematics* **2021**, *9*, 183. https://doi.org/10.3390/math9020183

Received: 23 December 2020
Accepted: 15 January 2021
Published: 18 January 2021

1. Introduction

The 2003 Nobel prize winning Ginzburg–Landau model in the field of physics is widely used in superconductors, superfluids, and condensation processes of Bose–Einstein type. As a low-temperature superconducting model [1], the Ginzburg–Landau model was first introduced by physicists Ginzburg and Landau in the 1950s. A wide variety of phenomena can be described by the Ginzburg–Landau equation starting from second-order phase transitions to nonlinear waves, and from Bose-Einstein condensation, superconductivity and superfluidity to strings in field theory and liquid crystals [2]. The concept of the fractional Ginzburg–Landau equation, which can be used to describe the dynamical processes in a medium with fractal dispersion was first derived by Tarasov in [3]. The variational Euler-Lagrange equation for fractal media was the generator of the fractional generalization of Ginzburg–Landau equation. A coupled Ginzberg-Landau system was used to describe a class of nonlinear optical fiber materials with active and passive coupled cores [4,5].

The well-posedness was discussed globally, and the long-time dynamics for the nonlinear complex Ginzburg–Landau equation involving fractional Laplacian was tackled in [6]. The dynamics and well-posedness of the coupled fractional Ginzburg–Landau equation, which describes a class of nonlinear optical fiber materials with active and passive coupled cores, was discussed in [7].

Motivated by their vast applications, numerical methods dealing with Ginzberg-Landau problems have gained attention due to the difficulty of obtaining exact solutions for the fractional order form of Ginzberg-Landau models. Armed by that, Galerkin spectral methods of Legendre type would be one of the most appropriate numerical methods for handling this kind of problem well. The Galerkin spectral method is a valuable tool for solving partial differential equations [8] and has been successfully applied to solving various types of fractional-order models [9–14].

The authors have already made some contributions to numerically solving various kinds of time-space fractional order problems based on the ideas of Legendre Galerkin spectral and finite difference schemes. In [15], semi-implicit spectral approximations were proposed to solve nonlinear time-space fractional diffusion–reaction equations with smooth and nonsmooth solutions. A combination of the Legendre Galerkin spectral approximation and the L1 difference approximation formulae over graded and uniform meshes was proposed. The work in [16] was concerned with a numerical treatment of nonlinear fractional Schrödinger equations with Riesz space-and Caputo time-fractional derivatives. The L1 finite difference approximation was used for the discretization of the Caputo fractional derivative and the Legendre-Galerkin spectral method was used for the spatial approximation. The contribution in [17] was devoted to coupled nonlinear time-space fractional Schrödinger equations with non-smooth solutions in the time direction. The method combined the L1 scheme with temporal nonuniform mesh and the Galerkin-Legendre spectral approximation. The convergence and the stability estimates were performed using energy estimates and discrete forms of Grönwall inequalities [18–20]. More recently, a numerical algorithm was proposed for the time–space fractional Ginzburg–Landau equation by a high-order difference/Galerkin spectral scheme. For the temporal approximation, the smooth Alikhanov difference formula was used to discretize the time fractional derivative of Caputo type, while for the spatial discretization, we hinged on the Legendre–Galerkin spectral method [21]. In [22], a graded mesh finite difference/Galerkin spectral method was used to numerically solve a coupled system of time and space fractional diffusion equations.

In this paper, we propose a high order Alikhanov Legendre-Galerkin spectral method for solving the following nonlinear coupled fractional Ginzburg–Landau equations:

$$ {}_0^C D_t^\beta \psi - (\nu_1 + i\eta_1)\, \frac{\partial^\alpha \psi}{\partial |x|^\alpha} + \big((k_1 + i\zeta_1)\, |\psi|^2 + (\varepsilon_1 + i\mu_1)\, |\phi|^2\big)\psi - \gamma_1\, \psi = 0, \quad x \in \Omega,\ t \in I, \tag{1a} $$

$$ {}_0^C D_t^\beta \phi - (\nu_2 + i\eta_2)\, \frac{\partial^\alpha \phi}{\partial |x|^\alpha} + \big((k_2 + i\zeta_2)\, |\psi|^2 + (\varepsilon_2 + i\mu_2)\, |\phi|^2\big)\phi - \gamma_2\, \phi = 0, \quad x \in \Omega,\ t \in I, \tag{1b} $$

with the initial conditions

$$ \psi(x,0) = \psi_0(x),\ \phi(x,0) = \phi_0(x),\ x \in \Omega, \tag{1c} $$

and the homogeneous boundary conditions

$$ \psi(a,t) = \psi(b,t) = \phi(a,t) = \phi(b,t) = 0,\ t \in I, \tag{1d} $$

such that $\Omega = (a,b) \subset \mathbb{R}$ and $I = (0,T] \subset \mathbb{R}$. The parameters ν_i, η_i, k_i, ζ_i, ε_i, μ_i and γ_i, $i = 1,\ 2$ are given real constants, and $\phi(x)$ is a given smooth function. The temporal fractional derivative is defined in Caputo sense [23]:

$$ {}_0^C D_t^\beta \Psi(x,t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_0^t \frac{1}{(t-r)^\beta} \frac{\partial \Psi(x,r)}{\partial r} \mathrm{d}r, & 0 < \beta < 1, \\ \dfrac{\partial \Psi(x,t)}{\partial t}, & \beta = 1. \end{cases} \tag{2} $$

The spatial fractional operator of Riesz type of order α with respect to $a \leq x \leq b$, namely, $\frac{\partial^\alpha \Psi}{\partial |x|^\alpha}$, is defined as [23]

$$\frac{\partial^\alpha \Psi}{\partial |x|^\alpha} = -c_\alpha \left({}_aD_x^\alpha \Psi(x,t) + {}_xD_b^\alpha \Psi(x,t) \right), \quad c_\alpha = \frac{1}{2\cos\frac{\pi\alpha}{2}}, \quad 1 < \alpha < 2,$$

where ${}_aD_x^\alpha \Psi(x,t)$ is the left-sided Riemann–Liouville derivative and ${}_xD_b^\alpha \Psi(x,t)$ is the right-sided Riemann–Liouville derivative of order α with respect to $x \in (a,b)$, defined as

$${}_aD_x^\alpha \Psi(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_a^x (x-\tau)^{n-1-\alpha} \Psi(\tau,t) d\tau, \tag{3}$$

$${}_xD_b^\alpha \Psi(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_x^b (\tau-x)^{n-1-\alpha} \Psi(\tau,t) d\tau. \tag{4}$$

There exists a wide variety of numerical methods which deal with space and/or fractional differential equations [24–31]. The coupled space fractional Ginzburg–Landau system was numerically investigated in [32]. A linearized semi-implicit difference scheme is proposed with unconditional stability and fourth order of convergence. In [33], a discrete difference scheme based on the implicit midpoint in time and a weighted and shifted Grünwald difference scheme with respect to space. The scheme is uniquely solvable, and the numerical solutions are bounded and unconditionally convergent. For the strongly coupled fractional Ginzburg-Landau system, a linearized three time level semi-implicit finite difference scheme in [34] was proposed to solve it. The difference scheme is unconditionally stable, fourth-order accurate in space, and second-order accurate in time.

The main concern of this work is to first design a combined numerical scheme for a coupled system (1) of Ginzburg-Landau with the time Caputo fractional derivative and the Riesz space fractional Laplacian operator. That scheme combines the Alikhanov $L2$-1_σ differentiation formula [35] with Legendre Galerkin spectral approximation. Error estimates and unconditional convergence of the proposed scheme based on discrete energy estimates are detailed here. Accordingly, the manuscript is organized as follows. The next section is devoted to some preliminaries. The numerical scheme and its implementation are illustrated in the third section. The fourth section focuses on the convergence analysis of the proposed scheme in both semi and full discretized styles. Some numerical experiments are done in the penultimate section while the manuscript ends with a section for conclusion and remarks.

2. Preliminaries

Some spaces of fractional derivatives are recalled below, see [36]. The notation $(\cdot,\cdot)_{0,\Omega}$ denotes the inner product on the space $L^2(\Omega)$ with the L^2-norm $\|\cdot\|_{0,\Omega}$ and the maximum norm $\|\cdot\|_\infty$. $C_0^\infty(\Omega)$ denotes the space of non-singular functions with compact support in Ω. $H^r(\Omega)$ and $H_0^r(\Omega)$ are Sobolev spaces with the norm $\|\cdot\|_{H^r}$ and semi-norm $|\cdot|_{H^r}$. We denote $\mathbb{P}_N(\Omega)$ the space of polynomials. The approximation space V_N^0 is defined as

$$V_N^0 = \mathbb{P}_N(\Omega) \cap H_0^1(\Omega).$$

Additionally, I_N is the interpolation operator of Legendre-Gauss-Lobatto type, $I_N : C(\bar{\Omega}) \to V_N$,

$$\Psi(x_k) = I_N \Psi(x_k) \in \mathbb{P}_N, \quad k = 0,1,\ldots,N.$$

Definition 1 (Left fractional derivative space)**.** *We define the semi-norm and the norm for* $\eta > 0$*, respectively as*

$$|\Psi|_{J_L^\eta(\Omega)} = \left\| {}_aD_x^\eta \Psi \right\|_{0,\Omega}, \quad \|\Psi\|_{J_L^\eta(\Omega)} = \left(|\Psi|^2_{J_L^\eta}(\Omega) + \|\Psi\|^2_{0,\Omega} \right)^{1/2},$$

such that J_L^η *is defined as the closure of* $C^\infty(\Omega)$ *with respect to* $\|\cdot\|_{J_L^\eta}$.

Definition 2 (Right fractional derivative space)**.** *We define the semi-norm and the norm for* $\eta > 0$*, respectively as*

$$|\Psi|_{J_R^\eta(\Omega)} = \left\| {}_xD_b^\eta \Psi \right\|_{0,\Omega}, \quad \|\Psi\|_{J_R^\eta(\Omega)} = \left(|\Psi|^2_{J_R^\eta(\Omega)} + \|\Psi\|^2_{0,\Omega} \right)^{1/2},$$

such that J_R^η *is defined as the closure of* $C^\infty(\Omega)$ *with respect to* $\|\cdot\|_{J_R^\eta}$.

Definition 3 (Symmetric fractional derivative space)**.** *We define the semi-norm and the norm for* $\eta \neq n - \frac{1}{2}$*,* $n \in \mathbb{N}$*, respectively as*

$$|\Psi|_{J_s^\eta(\Omega)} = \left| \left({}_aD_x^\eta \Psi, {}_xD_b^\eta \Psi \right)_{0,\Omega} \right|^{1/2}, \quad \|\Psi\|_{J_s^\eta(\Omega)} = \left(|\Psi|^2_{J_s^\eta(\Omega)} + \|\Psi\|^2_{0,\Omega} \right)^{1/2},$$

and denote J_s^η *as* $C^\infty(\Omega)$ *closure with respect to* $\|\cdot\|_{J_s^\eta}$.

Definition 4 (Fractional Sobolev space)**.** *The Sobolev space* $H^\eta(\Omega)$ *for* $\eta > 0$*, is given as*

$$H^\eta(\Omega) = \left\{ \Psi \in L^2(\Omega) \, : |\omega|^\eta \mathcal{F}(\tilde{\Psi}) \in L^2(\mathbb{R}) \right\},$$

endowed with the semi-norm and norm respectively as

$$|\ \Psi\ |_{H^\eta(\Omega)} = \|\ \ |\ \omega\ |^\eta\ \mathcal{F}(\tilde{\Psi})\|_{0,\mathbb{R}}, \quad \|\Psi\|_{H^\eta(\Omega)} = \left(|\ \Psi\ |^2_{H^\eta(\Omega)} + \|\Psi\|^2_{0,\Omega} \right)^{1/2},$$

such that $H_0^\eta(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *with respect to* $\|\cdot\|_{H^\eta(\Omega)}$*. Also,* $\mathcal{F}(\tilde{\Psi})$ *is the Fourier transformation of the function* $\tilde{\Psi}$ *and the zero extension of* Ψ *outside* Ω *denoted by* $\tilde{\Psi}$*.*

Lemma 1 (Adjoint property)**.** *By choosing* $1 < \eta < 2$*, then* $\forall\, \Psi \in H_0^\eta(\Omega)$ *and* $\nu \in H_0^{\eta/2}(\Omega)$*, we deduce*

$$\left({}_aD_x^\eta \Psi, \nu \right)_{0,\Omega} = \left({}_aD_x^{\eta/2} \Psi, {}_xD_b^{\eta/2} \nu \right)_{0,\Omega}, \quad \left({}_xD_b^\eta \Psi, \nu \right)_{0,\Omega} = \left({}_xD_b^{\eta/2} \Psi, {}_aD_x^{\eta/2} \nu \right)_{0,\Omega}.$$

3. Numerical Scheme

Here, the discretization of problem (1) is done by using the $L2$-1_σ approximation difference formula for the Caputo time fractional operator side by side to the Legendre-Galerkin spectral method for the Riesz spatial-fractional operator. A detailed implementation of the proposed scheme is proposed here.

3.1. Discretization

We partition the temporal domain I by $t_j = j\tau$, $j = 0, 1, \ldots, M$ with $\tau = T/M$. Denote $t_{j+\sigma} = (j+\sigma)\tau = \sigma t_{j+1} + (1-\sigma)t_j$, for $j = 0, 1, \ldots, M-1$. Take $\Psi^{j+\sigma} = \Psi^{j+\sigma}(\cdot) = \Psi(\cdot, t_{j+\sigma})$.

Definition 5. *Let* $0 < \beta < 1$ *and* $\sigma = 1 - \frac{\beta}{2}$*. Define*

$$a_s^{(\beta,\sigma)} = \begin{cases} \sigma^{1-\beta}, & s = 0, \\ (s+\sigma)^{1-\beta} - (s-1+\sigma)^{1-\beta}, & s \geq 1, \end{cases} \tag{5}$$

$$b_s^{(\beta,\sigma)} = \frac{1}{2-\beta}\left[(s+\sigma)^{2-\beta} - (s-1+\sigma)^{2-\beta}\right] - \frac{1}{2}\left[(s+\sigma)^{1-\beta} + (s-1+\sigma)^{1-\beta}\right], \quad s \geq 1, \tag{6}$$

and

$$C_s^{(j,\beta,\sigma)} = \begin{cases} a_0^{(\beta,\sigma)}, & s = j = 0, \\ a_0^{(\beta,\sigma)} + b_1^{(\beta,\sigma)}, & s = 0,\ j \geq 1, \\ a_s^{(\beta,\sigma)} + b_{s+1}^{(\beta,\sigma)} - b_s^{(\beta,\sigma)}, & 1 \leq s \leq j-1, \\ a_j^{(\beta,\sigma)} - b_j^{(\beta,\sigma)}, & 1 \leq s = j. \end{cases} \tag{7}$$

Lemma 2 (see [35])**.** *The high order Alikhanov L_2-1_σ difference formula under the assumption $\Psi(t) \in C^3[0, t_{j+1}]$, $0 \leq j \leq M-1$, formulated as*

$$_0D^{\beta}_{t_{j+\sigma}}\Psi = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{r=0}^{j} C_{j-r}^{(j,\beta,\sigma)} \delta_t \Psi^r + \mathcal{O}(\tau^{3-\beta}), \qquad 0 < \beta < 1, \tag{8}$$

where $\delta_t \Psi^r = \Psi^{r+1} - \Psi^r$.

It can be rewritten as

$$_0D^{\beta}_{t_{j+\sigma}}\Psi = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{r=0}^{j} d_r^{(j,\beta,\sigma)} \Psi^r + \mathcal{O}(\tau^{3-\beta}), \tag{9}$$

where $d_1^{(0,\beta,\sigma)} = -d_0^{(0,\beta,\sigma)} = \sigma^{1-\beta}\ \forall j = 0$, and $\forall j \geq 1$,

$$d_s^{(j,\beta,\sigma)} = \begin{cases} -C_j^{(j,\beta,\sigma)}, & s = 0, \\ C_{j-s+1}^{(j,\beta,\sigma)} - C_{j-s}^{(j,\beta,\sigma)}, & 1 \leq s \leq j, \\ C_0^{(j,\beta,\sigma)}, & s = j+1. \end{cases} \tag{10}$$

Definition 6. *Let $j \in \mathbb{Z}_{[0,M-1]}$, Alikhanov L2-$1_\sigma$ difference formula at the node $t_{j+\sigma}$ is defined as*

$$_0D^{\beta}_{\tau}\Psi^{j+\sigma} = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{r=0}^{j+1} d_r^{(j,\beta,\sigma)} \Psi^r, \qquad 0 < \beta < 1. \tag{11}$$

The following identity holds directly by Taylor's theorem.

Lemma 3. *The following identity holds:*

$$\Psi(\cdot, t_{j+\sigma}) = \sigma\Psi(\cdot, t_{j+1}) + (1-\sigma)\Psi(\cdot, t_j) + \mathcal{O}(\tau^2). \tag{12}$$

Starting from the $L2$-1_σ Formula (11) for the discretization of the time Caputo fractional derivative of (1a), this leads to

$$\begin{aligned} {}_0D^{\beta}_{\tau}\psi^{j+\sigma} - (\nu_1 + i\eta_1)\,\frac{\partial^\alpha \psi^{j+\sigma}}{\partial |x|^\alpha} + (k_1 + i\zeta_1)\,|\psi^{j+\sigma}|^2\psi^{j+\sigma} + \\ (\varepsilon_1 + i\mu_1)\,|\phi^{j+\sigma}|^2\psi^{j+\sigma} - \gamma_1\,\psi^{j+\sigma} = 0, \quad x \in \Omega, \end{aligned} \tag{13a}$$

$$\begin{aligned} {}_0D^{\beta}_{\tau}\phi^{j+\sigma} - (\nu_2 + i\eta_2)\,\frac{\partial^\alpha \phi^{j+\sigma}}{\partial |x|^\alpha} + (k_2 + i\zeta_2)\,|\psi^{j+\sigma}|^2\phi^{j+\sigma} \\ + (\varepsilon_2 + i\mu_2)\,|\phi^{j+\sigma}|^2\phi^{j+\sigma} - \gamma_2\,\phi^{j+\sigma} = 0, \quad x \in \Omega. \end{aligned} \tag{13b}$$

By aid of Lemmas 2 and 3, this semi-scheme is of second order accuracy. Let us introduce the parameters

$$\xi_{j,r}^{(\beta,\sigma)} = \left(\frac{d_{j+1}^{(j,\beta,\sigma)}}{\tau^\beta \Gamma(2-\beta)} - \gamma_r \sigma \right)^{-1},$$

$$\tilde{d}_{i,r}^{(j,\beta,\sigma)} = \begin{cases} \frac{\xi_{j,r}^{(\beta,\sigma)} d_i^{(j,\beta,\sigma)}}{\tau^\beta \Gamma(2-\beta)}, & 0 \le i \le j-1, \\ \frac{\xi_{j,r}^{(\beta,\sigma)} d_j^{(j,\beta,\sigma)}}{\tau^\beta \Gamma(2-\beta)} - \gamma_r (1-\sigma) \xi_{j,r}^{(\beta,\sigma)}, & i = j, \ r = 1,2. \end{cases}$$

The semi-scheme (13) has following equivalent form:

$$\begin{aligned} \psi^{j+1} - (v_1 + i\eta_1)\sigma \xi_{j,1}^{(\beta,\sigma,\gamma)} \frac{\partial^\alpha \psi^{j+1}}{\partial |x|^\alpha} &= (v_1 + i\eta_1)(1-\sigma)\xi_{j,1}^{(\beta,\sigma,\gamma)} \frac{\partial^\alpha \psi^j}{\partial |x|^\alpha} \\ &- \sum_{i=0}^{j} \tilde{d}_{j,1}^{(j,\beta,\sigma)} \psi^i - \sigma(k_1 + i\zeta_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left|\psi^{j+1}\right|^2 \psi^{j+1} \\ &- \sigma(\varepsilon_1 + i\mu_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left|\phi^{j+1}\right|^2 \psi^{j+1} - (1-\sigma)(k_1 + i\zeta_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left|\psi^j\right|^2 \psi^j \\ &- (1-\sigma)(\varepsilon_1 + i\mu_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left|\phi^j\right|^2 \psi^j, \end{aligned} \tag{14a}$$

$$\begin{aligned} \phi^{j+1} - (v_2 + i\eta_2)\sigma \xi_{j,2}^{(\beta,\sigma,\gamma)} \frac{\partial^\alpha \phi^{j+1}}{\partial |x|^\alpha} &= (v_2 + i\eta_2)(1-\sigma)\xi_{j,2}^{(\beta,\sigma,\gamma)} \frac{\partial^\alpha \phi^j}{\partial |x|^\alpha} \\ &- \sum_{i=0}^{j} \tilde{d}_{j,2}^{(j,\beta,\sigma)} \phi^i - \sigma(k_2 + i\zeta_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left|\psi^{j+1}\right|^2 \phi^{j+1} \\ &- \sigma(\varepsilon_2 + i\mu_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left|\phi^{j+1}\right|^2 \phi^{j+1} - (1-\sigma)(k_2 + i\zeta_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left|\psi^j\right|^2 \phi^j \\ &- (1-\sigma)(\varepsilon_2 + i\mu_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left|\phi^j\right|^2 \phi^j. \end{aligned} \tag{14b}$$

And so, the full discrete Alikhanov $L2$-1_σ Galerkin spectral scheme for (14) is to get ψ_N^{j+1}, $\phi_N^{j+1} \in V_N^0$, $j \ge 0$, $\forall \nu \in V_N^0$ such that

$$\begin{cases} \left(\psi_N^{j+1}, \nu\right) - (v_1 + i\eta_1)\sigma \xi_{j,1}^{(\beta,\sigma,\gamma)} \left(\frac{\partial^\alpha \psi_N^{j+1}}{\partial |x|^\alpha}, \nu\right) \\ \qquad = (v_1 + i\eta_1)(1-\sigma)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left(\frac{\partial^\alpha \psi_N^j}{\partial |x|^\alpha}, \nu\right) \\ \qquad - \sum_{i=0}^{j} \tilde{d}_{j,1}^{(j,\beta,\sigma)} \left(\psi_N^i, \nu\right) - \sigma(k_1 + i\zeta_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left(I_N \left|\psi_N^{j+1}\right|^2 \psi_N^{j+1}, \nu\right) \\ \qquad - \sigma(\varepsilon_1 + i\mu_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left(I_N \left|\phi_N^{j+1}\right|^2 \psi_N^{j+1}, \nu\right) \\ \qquad - (1-\sigma)(k_1 + i\zeta_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left(I_N \left|\psi_N^j\right|^2 \psi_N^j, \nu\right) \\ \qquad - (1-\sigma)(\varepsilon_1 + i\mu_1)\xi_{j,1}^{(\beta,\sigma,\gamma)} \left(I_N \left|\phi_N^j\right|^2 \psi_N^j, \nu\right), \\ \left(\phi_N^{j+1}, \nu\right) - (v_2 + i\eta_2)\sigma \xi_{j,2}^{(\beta,\sigma,\gamma)} \left(\frac{\partial^\alpha \phi_N^{j+1}}{\partial |x|^\alpha}, \nu\right) \\ \qquad = (v_2 + i\eta_2)(1-\sigma)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left(\frac{\partial^\alpha \phi_N^j}{\partial |x|^\alpha}, \nu\right) \\ \qquad - \sum_{i=0}^{j} \tilde{d}_{j,2}^{(j,\beta,\sigma)} \left(\phi_N^i, \nu\right) - \sigma(k_2 + i\zeta_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left(I_N \left|\psi_N^{j+1}\right|^2 \phi_N^{j+1}, \nu\right) \\ \qquad - \sigma(\varepsilon_2 + i\mu_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left(I_N \left|\phi_N^{j+1}\right|^2 \phi_N^{j+1}, \nu\right) \\ \qquad - (1-\sigma)(k_2 + i\zeta_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left(I_N \left|\psi_N^j\right|^2 \phi_N^j, \nu\right) \\ \qquad - (1-\sigma)(\varepsilon_2 + i\mu_2)\xi_{j,2}^{(\beta,\sigma,\gamma)} \left(I_N \left|\phi_N^j\right|^2 \phi_N^j, \nu\right), \\ \psi_N^0 = P_N \psi_0, \quad \phi_N^0 = P_N \phi_0, \end{cases} \tag{15}$$

where P_N is a suitable projection operator. Its related features are illustrated in Section 4.

3.2. Algorithmic Implementation

Via the hypergeometric function, Jacobi polynomials can be presented for α, $\beta > -1$ and $x \in (-1, 1)$ [8]:

$$J_i^{\alpha,\beta}(x) = \frac{(\alpha+1)_i}{i!}\,{}_2F_1\left(-i, \alpha+\beta+i+1; \alpha+1; \frac{1-x}{2}\right),\ x \in (-1,1),\ i \in \mathbb{N}, \tag{16}$$

such that the notation $(\cdot)_i$ represents the symbol of Pochhammer. Armed by (16), we get the equivalent three-term recurrence relation

$$\begin{aligned} J_0^{\alpha,\beta}(x) &= 1, \\ J_1^{\alpha,\beta}(x) &= \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \\ J_{i+1}^{\alpha,\beta}(x) &= \left(\hat{a}_i^{\alpha,\beta}x - \hat{b}_i^{\alpha,\beta}\right)J_i^{\alpha,\beta}(x) - \hat{c}_i^{\alpha,\beta}J_{i-1}^{\alpha,\beta}(x), \quad i \geq 1, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \hat{a}_i^{\alpha,\beta} &= \frac{(2i+\beta+\alpha+1)(2i+\beta+\alpha+2)}{2(i+1)(i+\beta+\alpha+1)}, \\ \hat{b}_i^{\alpha,\beta} &= \frac{(2i+\beta+\alpha+1)(\beta^2-\alpha^2)}{2(i+1)(i+\beta+\alpha+1)(2i+\beta+\alpha)}, \\ \hat{c}_i^{\alpha,\beta} &= \frac{(2i+\beta+\alpha+2)(i+\alpha)(i+\beta)}{(i+1)(i+\beta+\alpha+1)(2i+\beta+\alpha)}. \end{aligned} \tag{18}$$

The Legendre polynomial $L_i(x)$ is a special case of the Jacobi polynomial, this means

$$L_i(x) = J_i^{0,0}(x) = {}_2F_1\left(-i, i+1; 1; \frac{1-x}{2}\right). \tag{19}$$

The weight function which makes the orthogonality of Jacobi polynomials occur is given as $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, namely,

$$\int_{-1}^{1} J_i^{\alpha,\beta}(x)J_j^{\alpha,\beta}(x)\omega^{\alpha,\beta}(x)dx = \gamma_i^{\alpha,\beta}\delta_{ij}, \tag{20}$$

where δ_{ij} is the Dirac Delta symbol, and

$$\gamma_i^{\alpha,\beta} = \frac{2^{(\alpha+\beta+1)}\Gamma(1+i+\beta)\Gamma(1+i+\alpha)}{i!(\alpha+2i+\beta+1)\Gamma(\alpha+\beta+i+1)}. \tag{21}$$

Lemma 4 (see for example [37])**.** *For $\alpha > 0$, one has*

$$\begin{aligned} {}_{-1}D_{\hat{x}}^{\alpha}L_r(\hat{x}) &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(1+\hat{x})^{-\alpha}J_r^{\alpha,-\alpha}(\hat{x}), \quad \hat{x} \in [-1,1], \\ {}_{\hat{x}}D_1^{\alpha}L_r(\hat{x}) &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)}(1-\hat{x})^{-\alpha}J_r^{-\alpha,\alpha}(\hat{x}), \quad \hat{x} \in [-1,1]. \end{aligned} \tag{22}$$

We introduce the following rescale functions:

$$\wedge : [a,b] \to [-1,1] : x \mapsto \frac{2x-(a+b)}{b-a}$$

$$\wedge^{-1} : [-1,1] \to [a,b] : t \mapsto \frac{(b-a)t+a+b}{2}$$

and we write $\wedge(x)$ as $\hat{x}$. The basis functions selected for the spatial discretization are given by [9,38]:

$$\varphi_n(x) = L_n(\hat{x}) - L_{n+2}(\hat{x}) = \frac{2n+3}{2(n+1)}(1-\hat{x}^2)J_n^{1,1}(\hat{x}), \quad x \in [a,b]. \tag{23}$$

The function space V_N^0 can be specified as follows:

$$V_N^0 = \mathrm{span}\{\varphi_n(x), \quad n = 0,1,\ldots,N-2\}. \tag{24}$$

The approximate solutions ψ_N^{j+1} and ϕ_N^{j+1} are shown as

$$\psi_N^{j+1}(x) = \sum_{i=0}^{N-2} \hat{\psi}_i^{j+1}\varphi_i(x), \qquad \phi_N^{j+1}(x) = \sum_{i=0}^{N-2} \hat{\phi}_i^{j+1}\varphi_i(x), \tag{25}$$

where $\hat{\psi}_i^{j+1}$ and $\hat{\phi}_i^{j+1}$ are the unknown expansion coefficients to be determined. Choosing $v = \varphi_i$, $0 \le i \le N-2$, the matrix representation of the Alikhanov $L2$-1_σ Legendre-Galerkin spectral scheme has the following representation:

$$\begin{aligned}
&\Big[M + (v_1 + i\eta_1)\sigma c_\alpha \xi_{j,1}^{(\beta,\sigma,\gamma)}\Big(S + S^T\Big)\Big]\Psi^{j+1} = R_1^j \\
&\qquad - (k_1 + i\zeta_1)\sigma\xi_{j,1}^{(\beta,\sigma,\gamma)}H_{11}^{j+1} - (\varepsilon_1 + i\mu_1)\sigma\xi_{j,1}^{(\beta,\sigma,\gamma)}H_{12}^{j+1}, \\
&\Big[M + (v_2 + i\eta_2)\sigma c_\alpha \xi_{j,2}^{(\beta,\sigma,\gamma)}\Big(S + S^T\Big)\Big]\Phi^{j+1} = R_2^j \\
&\qquad - (k_2 + i\zeta_2)\sigma\xi_{j,2}^{(\beta,\sigma,\gamma)}H_{21}^{j+1} - (\varepsilon_2 + i\mu_2)\sigma\xi_{j,2}^{(\beta,\sigma,\gamma)}H_{22}^{j+1},
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&\Psi^j = (\hat{\psi}_0^j, \hat{\psi}_1^j, \ldots, \hat{\psi}_{N-2}^j)^T, \quad \Phi^j = (\hat{\phi}_0^j, \hat{\phi}_1^j, \ldots, \hat{\phi}_{N-2}^j)^T, \\
&s_{ij} = \int_\Omega {}_aD_x^{\frac{\alpha}{2}}\varphi_i(x)\,{}_xD_b^{\frac{\alpha}{2}}\varphi_j(x)dx, \qquad S = (s_{ij})_{i,j=0}^{N-2}, \\
&m_{ij} = \int_\Omega \varphi_i(x)\varphi_j(x)dx, \qquad M = (m_{ij})_{i,j=0}^{N-2}, \\
&h_{11,i}^j = \int_\Omega \varphi_i(x)I_N|\psi_N^j|^2\psi_N^j dx, \qquad H_{11}^j = (h_{11,0}^j, h_{11,1}^j, \ldots, h_{11,N-2}^j)^T, \\
&h_{12,i}^j = \int_\Omega \varphi_i(x)I_N|\phi_N^j|^2\psi_N^j dx, \qquad H_{12}^j = (h_{12,0}^j, h_{12,1}^j, \ldots, h_{12,N-2}^j)^T, \\
&h_{21,i}^j = \int_\Omega \varphi_i(x)I_N|\psi_N^j|^2\phi_N^j dx, \qquad H_{21}^j = (h_{21,0}^j, h_{21,1}^j, \ldots, h_{21,N-2}^j)^T, \\
&h_{22,i}^j = \int_\Omega \varphi_i(x)I_N|\phi_N^j|^2\phi_N^j dx, \qquad H_{22}^j = (h_{22,0}^j, h_{22,1}^j, \ldots, h_{22,N-2}^j)^T, \\
&K_1^j = \sum_{i=0}^{j} \tilde{d}_{i,1}^{(j,\beta,\sigma)}M\Psi^i, \qquad K_2^j = \sum_{i=0}^{j} \tilde{d}_{i,2}^{(j,\beta,\sigma)}M\Phi^i, \\
&R_1^j = -c_\alpha(v_1 + i\eta_1)(1-\sigma)\xi_{j,1}^{(\beta,\sigma,\gamma)}\Big(S + S^T\Big)\Psi^j - (k_1 + i\zeta_1)(1-\sigma)\xi_{j,1}^{(\beta,\sigma,\gamma)}H_{11}^j \\
&\qquad - (\varepsilon_1 + i\mu_1)(1-\sigma)\xi_{j,1}^{(\beta,\sigma,\gamma)}H_{12}^j - K_1^j, \\
&R_2^j = -c_\alpha(v_2 + i\eta_2)(1-\sigma)\xi_{j,2}^{(\beta,\sigma,\gamma)}\Big(S + S^T\Big)\Phi^j - (k_2 + i\zeta_2)(1-\sigma)\xi_{j,2}^{(\beta,\sigma,\gamma)}H_{21}^j \\
&\qquad - (\varepsilon_2 + i\mu_2)(1-\sigma)\xi_{j,2}^{(\beta,\sigma,\gamma)}H_{22}^j - K_2^j.
\end{aligned} \tag{27}$$

Lemma 5 (see [8,9])**.** *The elements of the stiffness matrix S are given by*

$$s_{ij} = a_i^j - a_i^{j+2} - a_{i+2}^j + a_{i+2}^{j+2}, \tag{28}$$

where

$$\begin{aligned} a_i^j &= \int_\Omega {}_aD_x^{\frac{\alpha}{2}} L_i(\hat{x})\, {}_xD_b^{\frac{\alpha}{2}} L_j(\hat{x}) dx \\ &= \left(\frac{b-a}{2}\right)^{1-\alpha} \frac{\Gamma(i+1)\Gamma(j+1)}{\Gamma(i-\frac{\alpha}{2}+1)\Gamma(j-\frac{\alpha}{2}+1)} \sum_{r=0}^{N} \varpi_r^{-\frac{\alpha}{2},-\frac{\alpha}{2}} J_i^{\frac{\alpha}{2},-\frac{\alpha}{2}}\left(x_r^{-\frac{\alpha}{2},-\frac{\alpha}{2}}\right) J_j^{-\frac{\alpha}{2},\frac{\alpha}{2}}\left(x_r^{-\frac{\alpha}{2},-\frac{\alpha}{2}}\right), \end{aligned} \tag{29}$$

and $\left\{x_r^{-\frac{\alpha}{2},-\frac{\alpha}{2}}, \varpi_r^{-\frac{\alpha}{2},-\frac{\alpha}{2}}\right\}_{r=0}^{N}$ *are Jacobi-Gauss points and their weights with weight function* $\omega^{-\frac{\alpha}{2},-\frac{\alpha}{2}}$*. The mass matrix M is symmetric and its nonzero elements are given as*

$$m_{ij} = m_{ji} = \begin{cases} \frac{b-a}{2j+1} + \frac{b-a}{2j+5}, & i = j, \\ -\frac{b-a}{2j+5}, & i = j+2. \end{cases} \tag{30}$$

Monitoring $H_{pq}^{j+1,r} = H_{pq}^{j+1}(\psi_N^{j+1,r}, \phi_N^{j+1,r})$, $p,\ q = 1,2$, $r \geq 0$, the linear system (26) can be solved by the iteration Algorithm 1:

Algorithm 1: Iterative algorithm for the problem (1).

Set $\Psi^{j+1,0} = \Psi^j$, $\psi_N^{j+1,0} = \sum_{i=0}^{N-2} \hat{\psi}_i^{j+1,0} \varphi_i(x)$, $\Phi^{j+1,0} = \Phi^j$, $\phi_N^{j+1,0} = \sum_{i=0}^{N-2} \hat{\phi}_i^{j+1,0} \varphi_i(x)$;

for $r = 0 : K$ **do**

Solve $\begin{cases} \left[M + (v_1 + i\eta_1)\sigma c_\alpha \xi_{j,1}^{(\beta,\sigma,\gamma)}(S+S^T)\right]\Psi^{j+1,r+1} = R_1^j \\ \qquad -(k_1 + i\zeta_1)\sigma\xi_{j,1}^{(\beta,\sigma,\gamma)} H_{11}^{j+1,r} - (\varepsilon_1 + i\mu_1)\sigma\xi_{j,1}^{(\beta,\sigma,\gamma)} H_{12}^{j+1,r}, \\ \left[M + (v_2 + i\eta_2)\sigma c_\alpha \xi_{j,2}^{(\beta,\sigma,\gamma)}(S+S^T)\right]\Phi^{j+1,r+1} = R_2^j \\ \qquad -(k_2 + i\zeta_2)\sigma\xi_{j,2}^{(\beta,\sigma,\gamma)} H_{21}^{j+1,r} - (\varepsilon_2 + i\mu_2)\sigma\xi_{j,2}^{(\beta,\sigma,\gamma)} H_{22}^{j+1,r}, \end{cases}$

to get $\Psi^{n,r+1}$ and $\Phi^{n,r+1}$;

Compute $\psi_N^{n,r+1} = \sum_{j=0}^{N-2} \hat{\psi}_j^{n,r+1} \varphi_j(x)$ and $\phi_N^{n,r+1} = \sum_{j=0}^{N-2} \hat{\phi}_j^{n,r+1} \varphi_j(x)$;

if $\left\|\psi_N^{n,r+1} - \psi_N^{n,r}\right\| \leq \epsilon$ & $\left\|\phi_N^{n,r+1} - \phi_N^{n,r}\right\| \leq \epsilon$ **then**

break;

end

end

Set $\Psi^n = \Psi^{n,r+1}$ and $\Phi^n = \Phi^{n,r+1}$.

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4. Convergence Analysis

We will present the convergence analysis of the Alikhanov $L2\text{-}1_\sigma$ Galerkin spectral scheme for the generalized fractional coupled Ginzburg–Landau system in both semi and full discretized forms. Any C represents a generic positive constant which can differ from one inequality to the another and is independent of τ, N and n.

Lemma 6 (see [10])**.** $\forall \Psi \in H_0^{\frac{\alpha}{2}}(\Omega) \cap H^s(\Omega)$, *there exists* P_N *such that:*

$$\|\Psi - P_N\Psi\| \leq CN^{-s}\|\Psi\|_s, \quad \alpha \neq \frac{3}{2}, \tag{31}$$

$$\|\Psi - P_N\Psi\| \leq CN^{\epsilon-s}\|\Psi\|_s, \quad \alpha = \frac{3}{2}, \quad 0 < \epsilon < \frac{1}{2}, \tag{32}$$

where ϵ *and* s *are real numbers satisfying* $s > \frac{\alpha}{2}$.

The interpolation operator I_N achieves the following property:

Lemma 7 (see [8]). *Suppose that $\Psi \in H^s(\Omega)$ $(s \geq 1)$, then*

$$\|\Psi - I_N\Psi\|_l \leq CN^{l-s}\|\Psi\|_s, \quad 0 \leq l \leq 1,$$

and $C > 0$ is a constant has no dependence on N.

Lemma 8 (see [35]). *Assume the existence of an absolute continuous function $\Psi(t)$ in $[0, T]$, then*

$$\Psi(t)\ {}_0D_t^{\beta}\Psi(t) \geq \frac{1}{2}D_t^{\beta}\Psi^2(t).$$

Lemma 9 (Grönwall inequality [23,39]). *Let $\Psi(t) \geq 0$ be a non-negative function is a local integrable function on $[0, +\infty]$ such that ${}_0D_t^{\beta}\Psi(t) \leq \lambda\Psi(t) + b$. Then, we have $\Psi(t) \leq \Psi_0 E_{\beta}(\lambda t^{\beta}) + bt^{\,\beta}E_{\beta,1+\beta}(\lambda t^{\beta})$, such that the Mittag-Leffler function $E_{\beta}(z)$ and the generalized Mittag-Leffler function $E_{\beta_1,\beta_2}(z)$ are defined by*

$$E_{\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\beta k)}, \quad E_{\beta_1,\beta_2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta_2+\beta_1 k)}, \quad \beta_1,\ \beta_2 > 0, \quad z \in \mathbb{C}.$$

Lemma 10 (see [23,39]). *For $0 < \beta_1 < 2$, and $\beta_2 \in \mathbb{R}$, we assume that μ such that $\pi\beta_1/2 < \mu < \min(\pi, \pi\beta_2)$. Then there exists a constant $C = C(\beta_1, \beta_2, \mu)$ such that $|E_{\beta_1,\beta_2}(z)| \leq \frac{C}{1+|z|}$, for $\mu \leq |arg(z)| \leq \pi$. In addition, if $\beta_1 \in (0,1)$, we have the following properties*

$$E_{\beta_1}(t) = E_{\beta_1,1}(t) > 0, \quad \frac{d}{dt}E_{\beta_1,\beta_1}(t) > 0.$$

Denote,

$$A(\psi, \Psi) = c_{\alpha}\left[\left({}_aD_x^{\alpha/2}\psi, {}_xD_b^{\alpha/2}\Psi\right) + \left({}_xD_b^{\alpha/2}\psi, {}_aD_x^{\alpha/2}\Psi\right)\right]. \tag{33}$$

For $1 < \alpha \leq 2$, the semi-norm and the norm are given as

$$|\Psi|_{\alpha/2} = \sqrt{A(\Psi, \Psi)}, \quad \|\Psi\|_{\alpha/2} = (\|\Psi\|^2 + |\Psi|^2_{\alpha/2})^{\frac{1}{2}}. \tag{34}$$

Also, if there exist positive constants C_1, C_2 such that for any u, $v \in H_0^{\alpha/2}(\Omega)$, we get

$$A(\psi, \Psi) \leq C_1\|\psi\|_{\alpha/2}\|\Psi\|_{\alpha/2}, \ A(\psi, \psi) \geq C_2\|\psi\|^2_{\alpha/2}. \tag{35}$$

The orthogonal projection operator $P_N : H_0^{\frac{\alpha}{2}}(\Omega) \to V_N^0$ satisfies

$$A(\psi - P_N\psi, \Psi) = 0, \quad \forall \Psi \in V_N^0.$$

The next lemma gives the relation between the L^p-norms and the fractional Sobolev norms [40].

Lemma 11. *If $0 \leq \mu_0 \leq \mu \leq 1$, $\ \frac{1}{2} - \frac{1}{P} < \mu_0 \leq 1$ and $2 \leq P \leq +\infty$, there exists $C_{\mu_0} > 0$, such that*

$$\|u\|_{L^P} \leq C\|u\|_{H^{\mu}}^{\frac{\mu_0}{\mu}}\|u\|^{1-\frac{\mu_0}{\mu}}.$$

Lemma 12 (see [35]). *Let $\Psi(t)$ be any function defined on Ω and $0 < \alpha < 1$. If $\Psi^{(\sigma)} = \sigma\Psi^{j+1} + (1-\sigma)\Psi^j$ then*

$$\Psi^{(\sigma)}\ {}_0D_{t_{j+\sigma}}^{\alpha}\Psi(t) \geq \frac{1}{2}\ {}_0D_{j+\sigma}^{\alpha}\Psi^2(t). \tag{36}$$

Lemma 13 ($L2$-1_σ discrete fractional form of Grönwall inequality [18,20]). *Suppose that the non-negative sequences* $\{\omega^j, g^j | j = 0, 1, 2, ...\}$ *satisfy* ${}_0D^\beta_\tau \omega^{j+\sigma} \le \lambda_1 \omega^{j+1} + \lambda_2 \omega^j + g^j$, *then there exists a positive constant* τ^* *such that*

$$\omega^{j+1} \le 2\left(\omega^0 + \frac{t_j^\beta}{\Gamma(1+\beta)} \max_{0\le j_0 \le n} g^{j_0}\right) E_\beta(2\lambda t_j^\beta), \tag{37}$$

whenever $\tau \le (\tau^*)^\beta = 1/(2\Gamma(2-\beta)\lambda_1)$ *and*

$$\lambda = \lambda_1 + \frac{\lambda_2}{c_0^{(\beta,\sigma)} - c_1^{(\beta,\sigma)}}. \tag{38}$$

4.1. Semi-Discrete form Convergence Analysis

Theorem 1. *Let* $\{0 < \beta < 1,\ 1 < \alpha < 2,\ s \ge 1\}$. *Assume that* $\{\psi, \phi\}$ *and* $\{\psi_N, \phi_N\}$ *are the solutions of* (1) *and* (13), *respectively, satisfying* $\{\psi, \phi\} \in H^1(I; H_0^{\frac{\alpha}{2}}(\Lambda) \cap H^s(\Lambda))$. *Then, we get*

$$\|\psi_N - \psi\| + \|\phi_N - \phi\| \le CN^{-s}, \quad \alpha \ne \frac{3}{2},$$

$$\|\psi_N - \psi\| + \|\phi_N - \phi\| \le CN^{\mu - s}, \quad \alpha = \frac{3}{2},\ 0 < \mu < \frac{1}{2}.$$

Proof. The variational formulation comes by taking the inner product of (1a) with $\mathbf{v}_1$,

$$\begin{aligned}({}_0D^\beta_t \psi, \mathbf{v}_1) - (\nu_1 + i\eta_1)\left(\frac{\partial^\alpha \psi}{\partial |x|^\alpha}, \mathbf{v}_1\right) + (k_1 + i\zeta_1)\left(|\psi|^2\psi, \mathbf{v}_1\right)\\ + (\varepsilon_1 + i\mu_1)\left(|\phi|^2\psi, \mathbf{v}_1\right) - \gamma\,(\psi, \mathbf{v}_1) = 0.\end{aligned} \tag{39}$$

Let $e = \psi - \psi_N$, $\zeta_e = \psi - P_N\psi$ and $\eta_e = P_N\psi - \psi_N$, we get $e = \zeta_e + \eta_e$. Also, let $E = \phi - \phi_N$, $\zeta_E = \phi - P_N\phi$ and $\eta_E = P_N\phi - \phi_N$, we get $E = \zeta_E + \eta_E$. We get the following estimate in the case of $\alpha \ne \frac{3}{2}$ by using Lemma 6,

$$\|e\| \le \|\zeta_e\| + \|\eta_e\| \le \mathcal{C}N^{-s}\|\psi\|_s + \|\eta_e\|, \tag{40}$$

$$\|E\| \le \|\zeta_E\| + \|\eta_E\| \le \mathcal{C}N^{-s}\|\phi\|_s + \|\eta_E\|. \tag{41}$$

Subtract (39) from (15), then we obtain

$$\begin{aligned}({}_0D^\beta_t e, \mathbf{v}_1) - (\nu_1 + i\eta_1)\left(\frac{\partial^\alpha e}{\partial |x|^\alpha}, \mathbf{v}_1\right) + (k_1 + i\zeta_1)\left(I_N|\psi_N|^2\psi_N - |\psi|^2\psi, \mathbf{v}_1\right)\\ + (\varepsilon_1 + i\mu_1)\left(I_N|\phi_N|^2\psi_N - |\phi|^2\psi, \mathbf{v}_1\right) - \gamma\,(e, \mathbf{v}_1) = 0.\end{aligned} \tag{42}$$

The orthogonality of P_N, yields

$$({}_0D^\beta_t e, \mathbf{v}_1) = (D^\beta_t \eta_e, \mathbf{v}_1), \tag{43}$$

$$({}_aD^\alpha_x e, \mathbf{v}_1) = (\zeta, {}_xD^\alpha_b \mathbf{v}_1) + ({}_aD^{\alpha/2}_x \eta_e, {}_xD^{\alpha/2}_b \mathbf{v}_1) = ({}_aD^{\alpha/2}_x \eta_e, {}_xD^{\alpha/2}_b \mathbf{v}_1). \tag{44}$$

Taking the inner product of (42) with η_e. Choosing the real part of the resulting equation, we get

$$\left({}_0D_t^{\beta} e, \eta_e \right) - \nu_1 \left(\frac{\partial^{\alpha} e}{\partial |x|^{\alpha}}, \eta_e \right) + Re \left[(k_1 + i\zeta_1) \left(I_N |\psi_N|^2 \psi_N - |\psi|^2 \psi, \eta_e \right) \right.$$
$$\left. + (\varepsilon_1 + i\mu_1) \left(I_N |\phi_N|^2 \psi_N - |\phi|^2 \psi, \eta_e \right) \right] - \gamma \left(e, \eta_e \right) = 0. \tag{45}$$

Invoking (33), (43) and (45). Using Lemma 8, we obtain

$$\frac{1}{2} \, {}_0D_t^{\beta} \|\eta_e\|^2 + \nu_1 C_2 \|\eta_e\|^2_{\alpha/2} + Re \left[(k_1 + i\zeta_1) \left(I_N |\psi_N|^2 \psi_N - |\psi|^2 \psi, \eta_e \right) \right.$$
$$\left. + (\varepsilon_1 + i\mu_1) \left(I_N |\phi_N|^2 \psi_N - |\phi|^2 \psi, \eta_e \right) \right] = \gamma \, \|\eta_e\|^2. \tag{46}$$

Define $G(\psi) = |\psi|^2 \psi$, and by the use of Cauchy-Schwarz inequality, we deduce

$$(I_N G(\psi_N) - G(\psi), \eta_e) \leq \frac{1}{2} \|I_N G(\psi_N) - G(\psi)\|^2 + \frac{1}{2} \|\eta_e\|^2,$$

$$\left(I_N |\phi_N|^2 \psi_N - |\phi|^2 \psi, \eta_e \right) \leq \frac{1}{2} \left\| I_N (|\phi_N|^2 \psi_N) - |\phi|^2 \psi \right\|^2 + \frac{1}{2} \|\eta_e\|^2.$$

By Lemma 7, we have

$$\|I_N G(\psi_N) - G(\psi)\| \leq \|I_N (G(\psi_N) - G(\psi))\| + \|I_N G(\psi) - G(\psi)\| \tag{47}$$
$$\leq \mathcal{C} \|G(\psi_N) - G(\psi)\| + C N^{-s} \|\psi\|_s. \tag{48}$$

We get as in [40] depending on Lemma 11 that

$$\|I_N G(\psi_N) - G(\psi)\| \leq \mathcal{C} N^{-s} \|\psi\|_s + \|\eta_e\|. \tag{49}$$

Also,

$$\begin{aligned} \left\| I_N (|\phi_N|^2 \psi_N) - |\phi|^2 \psi \right\| &= \left\| |\phi|^2 (I_N \psi_N - \psi) + \left(I_N |\phi_N|^2 - |\phi|^2 \right) I_N \psi_N \right\| \\ &\leq \|\phi\|_s^2 (N^{-s} \|\psi\|_s + \|\eta_e\|) + c_1 (\|\phi\|_s + c_2)(N^{-s} \|\phi\|_s + \|\eta_E\|) \\ &\leq \hat{C} (N^{-s} + \|\eta_e\| + \|\eta_E\|). \end{aligned} \tag{50}$$

Hence, we get the following estimate

$${}_0D_t^{\beta} \|\eta_e\|^2 + C_2 \nu_1 \|\eta_e\|^2_{\alpha/2} \leq \tilde{\mathcal{C}} \left(N^{-2s} + \|\eta_e\|^2 + \|\eta_E\|^2 \right). \tag{51}$$

Simultaneously, by taking the inner product of each part of (1b) with $\underset{\approx}{\zeta}_2$ and following the same steps as before, we also get

$${}_0D_t^{\beta} \|\eta_E\|^2 + C_2 \nu_1 \|\eta_E\|^2_{\alpha/2} \leq \bar{\mathcal{C}} \left(N^{-2s} + \|\eta_e\|^2 + \|\eta_E\|^2 \right). \tag{52}$$

Then adding (51) and (52) leads to

$${}_0D_t^{\beta} \left(\|\eta_e\|^2 + \|\eta_E\|^2 \right) \leq 2 \max\{\bar{\mathcal{C}}, \tilde{\mathcal{C}}\} \left(N^{-2s} + (\|\eta_e\| + \|\eta_E\|)^2 \right). \tag{53}$$

By Lemma 9, we obtain

$$\|\eta_e\|^2 + \|\eta_E\|^2 \le \left(\|\eta_{10}\| + \|\eta_{20}\|\right)^2 E_\beta(2\max\{C,\tilde{C}\}t^\beta) + 2\max\{\bar{C},\tilde{C}\}N^{-2s}t^\beta E_{\beta,1+\beta}(2\max\{\bar{C},\tilde{C}\}t^\beta). \tag{54}$$

Lemma 10 implies now that $t^\beta E_{\beta,1+\beta}(\tilde{C}t^\beta) \le C$. Finally we can see that $\|\eta_e\|^2 + \|\eta_E\|^2 \le 2C\max\{C,\tilde{C}\}N^{-2s}$. The other inequality of the conclusion can be achieved in a similar fashion if $\alpha = \frac{3}{2}$ and $0 < \mu < \frac{1}{2}$. $\square$

4.2. Full-Discrete form Convergence Analysis

Theorem 2 (Convergence of the uniform $L2$-1_σ – Galerkin spectral scheme)**.** *Let $\{\psi,\phi\}$ and $\{\psi_N^n,\phi_N^n\}$ be solutions of (1) and (13), respectively, and suppose that the unique solution $\{\psi,\phi\} \in L^\infty([0,T;H_0^{\frac{\alpha}{2}}(\Omega)\cap H^s(\Omega))$ is sufficiently regular in temporal and spatial directions and $\frac{\partial^\beta\psi}{\partial t^\beta}, \frac{\partial^\beta\phi}{\partial t^\beta} \in L^\infty([0,T;H_0^{\frac{\alpha}{2}}(\Omega)\cap H^s(\Omega))$. Then, a positive constant τ^* is existed such that when $0<\tau\le\tau^*$, the Galerkin spectral scheme (13a)–(13b) admits a unique solution $\{\psi_N^n,\phi_N^n\}$ satisfying*

$$\|\psi_N^n - \psi(x,t_n)\| + \|\phi_N^n - \phi(x,t_n)\| \le C\left(\tau^2 + N^{-s}\right), \qquad \text{if } \alpha \ne \tfrac{3}{2}, \tag{55}$$

$$\|\psi_N^n - \psi(x,t_n)\| + \|\phi_N^n - \phi(x,t_n)\| \le C\left(\tau^2 + N^{\mu-s}\right), \qquad \text{if } \alpha = \tfrac{3}{2} \text{ and } 0<\mu<\frac{1}{2}, \tag{56}$$

such that the C is a positive has no dependence on n, τ and N.

Proof. The next variational formula is derived by taking the inner product of (1a) with $\mathbf{v}_1$,

$$(D_t^\beta\psi^{j+\sigma},\mathbf{v}_1) - (\nu_1+i\eta_1)\left(\frac{\partial^\alpha\psi^{j+\sigma}}{\partial|x|^\alpha},\mathbf{v}_1\right) + (k_1+i\zeta_1)\left(|\psi^{j+\sigma}|^2\psi^{j+\sigma},\mathbf{v}_1\right) + (\varepsilon_1+i\mu_1)\left(|\phi^{j+\sigma}|^2\psi^{j+\sigma},\mathbf{v}_1\right) - \gamma\,(\psi^{j+\sigma},\mathbf{v}_1) + (O(\tau^2),\mathbf{v}_1) = 0, \tag{57}$$

Let $e=\psi-\psi_N$, $\zeta_e=\psi-P_N\psi$ and $\eta_e=P_N\psi-\psi_N$, we get $e^{j+\sigma}=\zeta_e^{j+\sigma}+\eta_e^{j+\sigma}$. Also, let $E=\phi-\phi_N$, $\zeta_E=\phi-P_N\phi$ and $\eta_E=P_N\phi-\phi_N$, we get $E^{j+\sigma}=\zeta_E^{j+\sigma}+\eta_E^{j+\sigma}$. Using Lemma 6, in case of $\alpha\ne\frac{3}{2}$, we get

$$\left\|e^{j+\sigma}\right\| \le \left\|\zeta_e^{j+\sigma}\right\| + \left\|\eta_e^{j+\sigma}\right\| \le CN^{-s}\left\|\psi^{j+\sigma}\right\|_s + \left\|\eta_e^{j+\sigma}\right\|, \tag{58}$$

$$\left\|E^{j+\sigma}\right\| \le \left\|\zeta_E^{j+\sigma}\right\| + \left\|\eta_E^{j+\sigma}\right\| \le CN^{-s}\left\|\phi^{j+\sigma}\right\|_s + \left\|\eta_E^{j+\sigma}\right\|. \tag{59}$$

Subtract (39) from (15), then we obtain

$$(D_t^\beta e^{j+\sigma},\mathbf{v}_1) - (\nu_1+i\eta_1)\left(\frac{\partial^\alpha e^{j+\sigma}}{\partial|x|^\alpha},\mathbf{v}_1\right) \tag{60}$$

$$+ (k_1+i\zeta_1)\left(I_N|\psi_N^{j+\sigma}|^2\psi_N^{j+\sigma} - |\psi^{j+\sigma}|^2\psi^{j+\sigma},\mathbf{v}_1\right)$$

$$+ (\varepsilon_1+i\mu_1)\left(I_N|\phi_N^{j+\sigma}|^2\psi_N^{j+\sigma} - |\phi^{j+\sigma}|^2\psi^{j+\sigma},\mathbf{v}_1\right) \tag{61}$$

$$-\gamma\,(e^{j+\sigma},\mathbf{v}_1) + (O(\tau^2),\mathbf{v}_1) = 0. \tag{62}$$

The orthogonality of the operator P_N, causes

$$(D_t^\beta e^{j+\sigma},\mathbf{v}_1) = (D_t^\beta\eta_e^{j+\sigma},\mathbf{v}_1), \tag{63}$$

$$({}_aD_x^\alpha e^{j+\sigma},\mathbf{v}_1) = (\zeta^{j+\sigma},{}_xD_b^\alpha\mathbf{v}_1) + ({}_aD_x^{\alpha/2}\eta_e^{j+\sigma},{}_xD_b^{\alpha/2}\mathbf{v}_1) = ({}_aD_x^{\alpha/2}\eta_e^{j+\sigma},{}_xD_b^{\alpha/2}\mathbf{v}_1). \tag{64}$$

After taking the inner product of (42) with $\eta_e^{j+\sigma}$, choose the real part of the resulting equation to obtain

$$
\begin{aligned}
&(D_t^{\beta} e^{j+\sigma}, \eta_e^{j+\sigma}) - \nu_1 \left(\frac{\partial^{\alpha} e^{j+\sigma}}{\partial |x|^{\alpha}}, \eta_e^{j+\sigma} \right) \\
&+ Re\Big[(k_1 + i\zeta_1) \left(I_N |\psi_N^{j+\sigma}|^2 \psi_N^{j+\sigma} - |\psi^{j+\sigma}|^2 \psi^{j+\sigma}, \eta_e^{j+\sigma} \right) \\
&\quad + (\varepsilon_1 + i\mu_1) \Big(I_N |\phi_N^{j+\sigma}|^2 \psi_N^{j+\sigma} - |\phi^{j+\sigma}|^2 \psi^{j+\sigma}, \eta_e^{j+\sigma} \Big) \Big] \\
&\quad - \gamma\, (e^{j+\sigma}, \eta_e^{j+\sigma}) + (\mathcal{O}(\tau^2), \eta_e^{j+\sigma}) = 0.
\end{aligned}
\tag{65}
$$

Invoking (33), (63) and (65). Using Lemma 8, we obtain

$$
\begin{aligned}
&\frac{1}{2} D_t^{\beta} \left\| \eta_e^{j+\sigma} \right\|^2 + \nu_1 C_2 \left\| \eta_e^{j+\sigma} \right\|_{\alpha/2}^2 \\
&+ Re\Big[(k_1 + i\zeta_1) \left(I_N |\psi_N^{j+\sigma}|^2 \psi_N^{j+\sigma} - |\psi^{j+\sigma}|^2 \psi^{j+\sigma}, \eta_e^{j+\sigma} \right) \\
&\quad + (\varepsilon_1 + i\mu_1) \Big(I_N |\phi_N^{j+\sigma}|^2 \psi_N^{j+\sigma} - |\phi^{j+\sigma}|^2 \psi^{j+\sigma}, \eta_e^{j+\sigma} \Big) \Big] \\
&\quad + (O(\tau^2), \eta_e^{j+\sigma}) = \gamma \left\| \eta_e^{j+\sigma} \right\|^2.
\end{aligned}
\tag{66}
$$

Proceeding as in the proof of Theorem 1, we define $G(\psi^{j+\sigma}) = |\psi^{j+\sigma}|^2 \psi^{j+\sigma}$, and using the Cauchy-Schwarz inequality, we get

$$
(I_N G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma}), \eta_e^{j+\sigma}) \le \frac{1}{2} \left\| I_N G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma}) \right\|^2 + \frac{1}{2} \|\eta_e\|^2,
$$

$$
\Big(I_N |\phi_N^{j+\sigma}|^2 \psi_N^{j+\sigma} - |\phi^{j+\sigma}|^2 \psi^{j+\sigma}, \eta_e^{j+\sigma} \Big) \le \frac{1}{2} \left\| I_N (|\phi_N^{j+\sigma}|^2 \psi_N^{j+\sigma}) - |\phi^{j+\sigma}|^2 \psi^{j+\sigma} \right\|^2 + \frac{1}{2} \left\| \eta_e^{j+\sigma} \right\|^2.
$$

By Lemma 7, we obtain

$$
\left\| I_N G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma}) \right\| \le \left\| I_N (G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma})) \right\| \tag{67}
$$

$$
+ \left\| I_N G(\psi^{j+\sigma}) - G(\psi^{j+\sigma}) \right\| \tag{68}
$$

$$
\le C \left\| G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma}) \right\| + C N^{-s} \left\| \psi^{j+\sigma} \right\|_s. \tag{69}
$$

We get as in [40] depending on Lemma 11 that

$$
\left\| I_N G(\psi_N^{j+\sigma}) - G(\psi^{j+\sigma}) \right\| \le C N^{-s} \left\| \psi^{j+\sigma} \right\|_s + \left\| \eta_e^{j+\sigma} \right\|. \tag{70}
$$

Also,

$$
\begin{aligned}
&\left\| I_N (|\phi_N^{j+\sigma}|^2 \psi_N^{j+\sigma}) - |\phi^{j+\sigma}|^2 \psi^{j+\sigma} \right\| \\
&= \left\| |\phi^{j+\sigma}|^2 \left(I_N \psi_N^{j+\sigma} - \psi^{j+\sigma} \right) + \left(I_N |\phi_N^{j+\sigma}|^2 - |\phi^{j+\sigma}|^2 \right) I_N \psi_N^{j+\sigma} \right\| \\
&\le \left\| \phi^{j+\sigma} \right\|_s^2 \left(N^{-s} \left\| \psi^{j+\sigma} \right\|_s + \left\| \eta_e^{j+\sigma} \right\| \right) \\
&\quad + c_1 \left(\left\| \phi^{j+\sigma} \right\|_s + c_2 \right) \left(N^{-s} \left\| \phi^{j+\sigma} \right\|_s + \left\| \eta_E^{j+\sigma} \right\| \right) \\
&\le \hat{C} \left(N^{-s} + \left\| \eta_e^{j+\sigma} \right\| + \left\| \eta_E^{j+\sigma} \right\| \right).
\end{aligned}
$$

The next estimate flows after some manipulations,

$$D_t^{\beta}\left\|\eta_e^{j+\sigma}\right\|^2 \le \tilde{C}\left(N^{-2s} + \left\|\eta_e^{j+1}\right\|^2 + \left\|\eta_e^{j}\right\|^2 + \left\|\eta_E^{j+1}\right\|^2 + \left\|\eta_E^{j}\right\|^2 + \tau^4\right). \tag{71}$$

Simultaneously,

$$D_t^{\beta}\left\|\eta_E^{j+\sigma}\right\|^2 \le \bar{C}\left(N^{-2s} + \left\|\eta_e^{j+1}\right\|^2 + \left\|\eta_e^{j}\right\|^2 + \left\|\eta_E^{j+1}\right\|^2 + \left\|\eta_E^{j}\right\|^2 + \tau^4\right). \tag{72}$$

Adding (71) to (72) and applying the $L2\text{-}1_{\sigma}$ Discrete fractional form of Grönwall inequality in Lemma 13, then the final result (55) is achieved directly. Similarly, we can get the result (56) when $\alpha = \frac{3}{2}$. Then the proof is fulfilled. □

5. Numerical Experiments

In this section, we provide two numerical examples to validate the analysis and the performance of the present scheme for the time-space fractional Ginzburg–Landau equations. All computations and visualizations have been carried out using Mathematica 12.1 on a personal computer with 12 GB memory and 2.3 GHz speed. Moreover, the spatial and the temporal convergence orders are computed using the following formulae:

$$\text{Order} = \begin{cases} \frac{\ln(\|e(M,K_1)\|/\|e(M,K_2)\|)}{\ln(K_1/K_2)}, & \text{in time,} \\ \frac{\ln(\|e(M_1,K)\|/\|e(M_2,K)\|)}{\ln(M_1/M_2)}, & \text{in space,} \end{cases}$$

where $M_1 \neq M_2$, $K_1 \neq K_2$ and

$$\text{Error} = e(M,K) = \max_{1\le n\le K}\left(\left\|\psi_{1,M}^n - \psi_1\right\| + \left\|\psi_{2,M}^n - \psi_2\right\|\right).$$

Example 1 (Convergence test)**.** *Consider the following nonlinear coupled system of fractional Ginzburg–Landau equations to test the accuracy of the proposed scheme:*

$${}_0^C D_t^{\beta}\psi - (1+i)\,\frac{\partial^{\alpha}\psi}{\partial|x|^{\alpha}} + \left((\tfrac{1}{2}+i)\,|\psi|^2 + i\,|\phi|^2\right)\psi - \psi = f_1(x,t), \quad x \in (0,1],\ t \in (0,1), \tag{73}$$

$${}_0^C D_t^{\beta}\phi - (1+i)\,\frac{\partial^{\alpha}\phi}{\partial|x|^{\alpha}} + \left(i\,|\psi|^2 + (\tfrac{1}{2}+i)\,|\phi|^2\right)\phi - \phi = f_2(x,t), \quad x \in (0,1),\ t \in (0,1], \tag{74}$$

with the homogeneous boundary conditions

$$\psi(a,t) = \psi(b,t) = \phi(a,t) = \phi(b,t) = 0,\ t \in I. \tag{75}$$

The initial conditions and the source terms $f_1(x,t)$ and $f_2(x,t)$ are determined by the exact solutions

$$\psi(x,t) = t^{3/2}x^2(1-x)^2, \quad \phi(x,t) = t^{7/3}x^2(1-x)^2.$$

Tables 1 and 2 list the L^2-errors and corresponding convergence orders with $\alpha = \beta + 1 = 1.2, 1.5, 1.8$ and $N = 100$ for ϕ and ψ, respectively. We can see that these results confirm the second-order convergence in time. The convergence orders in space a are depicted for different values of α and β at $M = 1600$ in Figures 1 and 2. All the convergence results are in agreement with the theoretical results.

Table 1. The L^2-errors and the convergence order of ϕ versus M and $\alpha = \beta + 1$ with $N = 100$ for example 1.

M	$\alpha = \beta + 1 = 1.2$		$\alpha = \beta + 1 = 1.5$		$\alpha = \beta + 1 = 1.8$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
100	5.046×10^{-7}	$--$	1.090×10^{-6}	$--$	1.449×10^{-6}	$--$
200	1.262×10^{-7}	1.999	2.726×10^{-7}	1.999	3.625×10^{-7}	1.999
400	3.164×10^{-8}	1.997	6.833×10^{-8}	1.996	9.093×10^{-8}	1.995
800	8.006×10^{-9}	1.983	1.734×10^{-8}	1.978	2.342×10^{-8}	1.957
1600	2.183×10^{-9}	1.874	4.871×10^{-9}	1.832	7.708×10^{-9}	1.603

Table 2. The L^2-errors and the convergence order of ψ versus M and $\alpha = \beta + 1$ with $N = 100$ for example 1.

M	$\alpha = \beta + 1 = 1.2$		$\alpha = \beta + 1 = 1.5$		$\alpha = \beta + 1 = 1.8$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
100	6.035×10^{-7}	$--$	1.300×10^{-6}	$--$	1.731×10^{-6}	$--$
200	1.510×10^{-7}	1.999	3.250×10^{-7}	1.999	4.329×10^{-7}	2.000
400	3.783×10^{-8}	1.997	8.143×10^{-8}	1.997	1.085×10^{-7}	1.996
800	9.549×10^{-9}	1.986	2.060×10^{-8}	1.983	2.773×10^{-8}	1.968
1600	2.551×10^{-9}	1.904	5.629×10^{-9}	1.871	8.578×10^{-9}	1.693

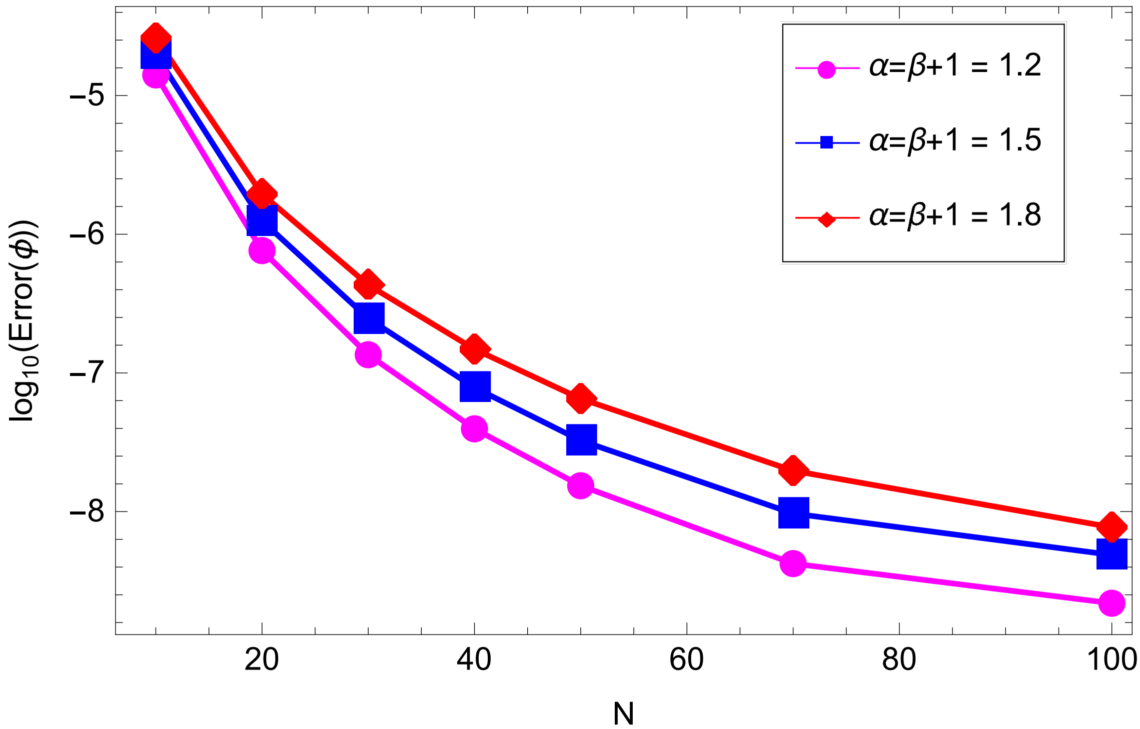

Figure 1. Convergence order of ϕ in space for different values of α and β at $M = 1600$.

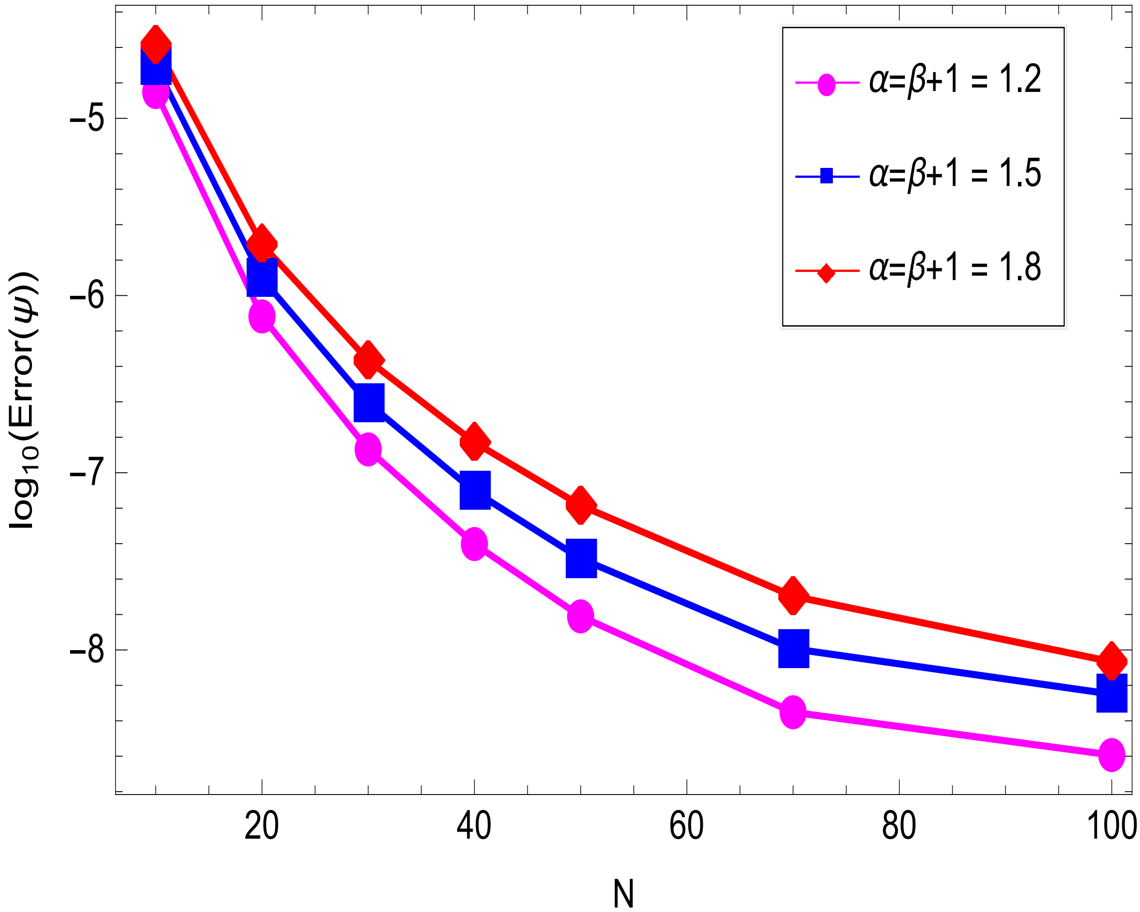

Figure 2. Convergence order of ψ in space for different values of α and β at $M = 1600$.

Example 2 (Model output)**.** *Consider the case of the coupled fractional Ginzburg–Landau Equations* (1) *with the initial values*

$$\psi(x, 0) = sech(x + D_0)\exp(-iV_0 x),$$

and

$$\phi(x, 0) = sech(x - D_0)\exp(-iV_0 x).$$

Henceforth, we take $D_0 = 2$, $V_0 = 3$, $x \in [-10, 10]$.

In this test, we select $\nu_1 = \nu_2 = \eta_1 = \eta_2 = \zeta_1 = \zeta_2 = \mu_1 = \mu_2 = 1$, $k_1 = k_2 = 1/2$ *and* $\varepsilon_1 = \varepsilon_2 = 0$*. Moreover, we will set the computational parameter* $N = 100$ *and* $M = 500$.

Figures 3 *and* 4 *display the numerical solutions for different* α *and* β *according to Example 2. We observe that, the fractional parameters* α *and* β *will dramatically affect the shape of the soliton, which is completely different from the classical case and shows the nonlocal character of the Caputo fractional derivative and fractional Laplacian. We find from these two figures also that the parameters* γ_1 *and* γ_2 *dramatically influence on the wave-shape. It is also observed that the numerical solutions decay fast with time evolution especially when the parameters* γ_1 *and* γ_2 *become more smaller and also when the fractional orders become more close to the integer orders, which means* α *comes close to* 2 *and* β *be closer to* 1*. These results seems to be in a good agreement with those appeared in* ([33], *Example 3).*

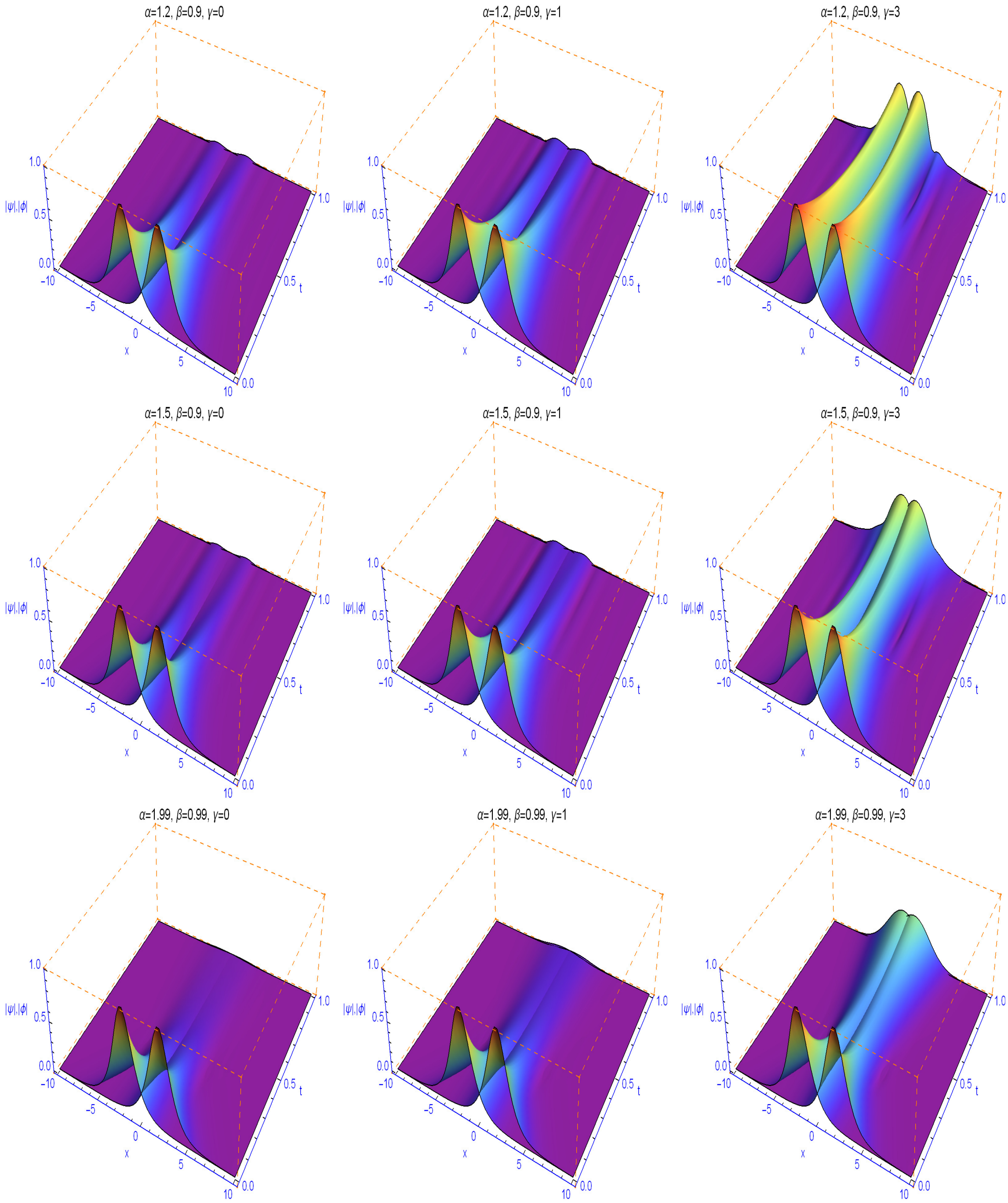

Figure 3. The evolution of $|\psi_N|$ and $|\phi_N|$ for different values of $\gamma_1 = \gamma_2 = \gamma = 0, 1, 3$. The first row is presented at $\alpha = 1.2$ and $\beta = 0.9$. The second row is presented at $\alpha = 1.5$ and $\beta = 0.9$. The third row is done at $\alpha = 1.99$ and $\beta = 0.99$.

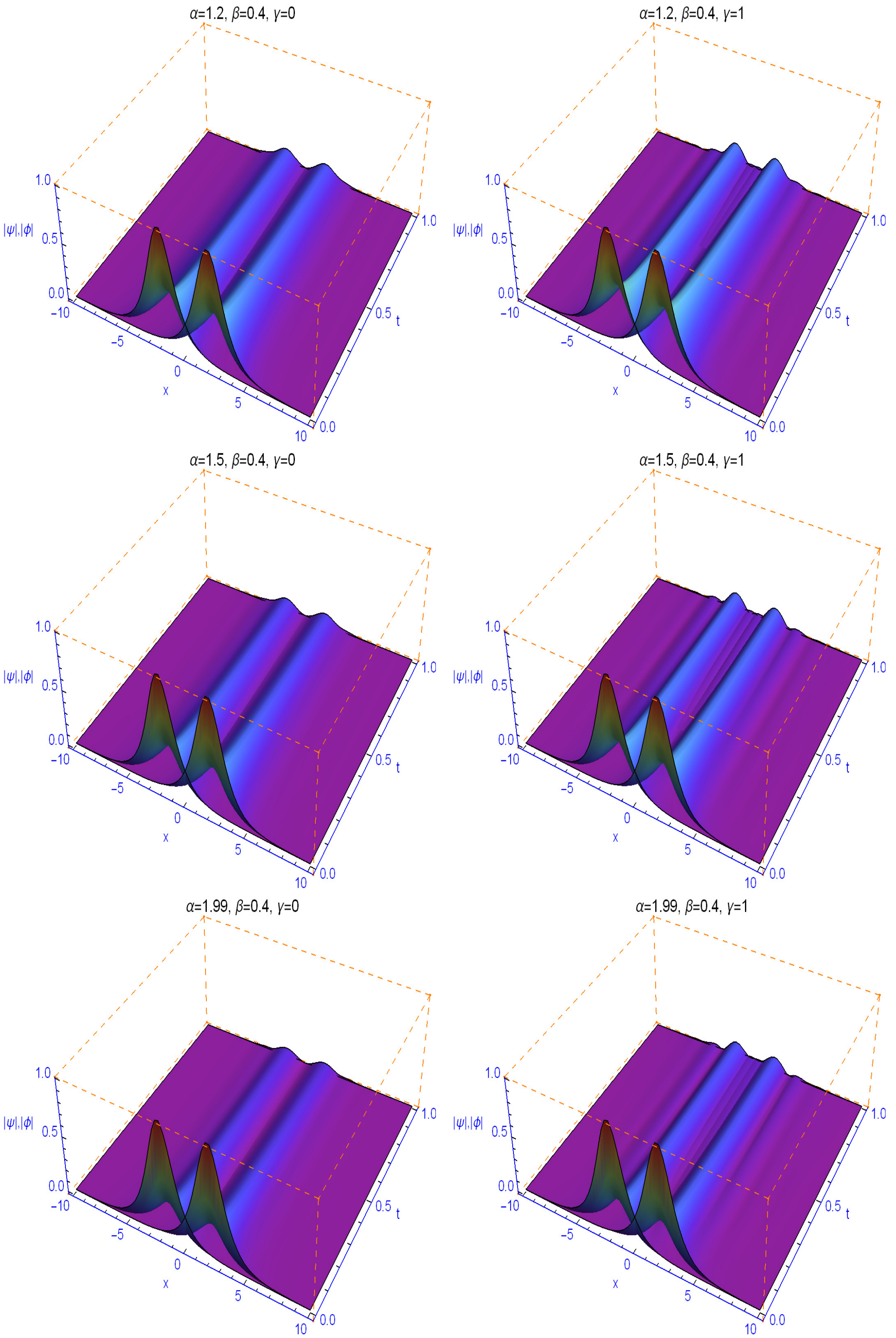

Figure 4. The evolution of $|\psi_N|$ and $|\phi_N|$ for different values of $\gamma_1 = \gamma_2 = \gamma = 0, 1$. The first row is presented at $\alpha = 1.2$ and $\beta = 0.4$. The second row is presented at $\alpha = 1.5$ and $\beta = 0.4$. The third row is done at $\alpha = 1.99$ and $\beta = 0.4$.

6. Conclusions

In this work, we provided an efficient high order numerical scheme for the generalized fractional coupled Ginzburg–Landau system. This scheme has a second order of convergence with respect to time and spectral accuracy with respect to space. The convergence analysis of the scheme shows the unconditional convergence of its approximate solution. An algorithmic easy implementation of the scheme is also provided to facilitate its numerical application. The paper ends with a numerical example, it shows the agreement between the theoretical results and numerical ones. Finally, we need to clarify the following issues and present some future work:

- Our proposed high order hybrid numerical scheme is a linearized scheme of second order of convergence with respect to time inspite of the nonlinearity of the problem under consideration. The spectral accuracy is achieved due to the use of Galerkin Legendre approximation. Up to our knowledge, it is the first time that scheme is used to solve that kind of problems, especially noting the appearance of time and space fractional derivatives in the model under study. Unconditional convergence and stability of that scheme is secured, which means the error estimates of the numerical model has no dependence on time and spatial steps. This work reflects the possibility of that kind of schemes to be extended to deal with success with the singularity near the initial values of time fractional Caputo operators appearing in the generalized Ginzburg–Landau system. The latter can be secured by using nonuniform Alikhanov schemes combined with Legendre Galerkin spectral and it would be a near future plan for us.
- Due to the intrinsically nonlocal property and historical dependence of the fractional derivative, numerical applications of the numerical methods are always time-consuming. Therefore, fast schemes based on local approximations [41,42] can be implemented to avoid the high computational costs coming from the prehistory feature of spatial fractional order operators. Fast L1 and Fast Alikhanov formulas of the Caputo derivative which are based on the sum of exponentials can be used to to reduce the huge storage and computational cost [43,44]. Invoking these approaches to reduce the computational cost of finite difference/Galerkin spectral methods would be a target of our new works in the near future.

Author Contributions: All the authors contributed as follows. Conceptualization and methodology, M.A.Z., R.H.D.S. and A.S.H.; software and validation, R.H.D.S., A.S.H. and M.A.Z.; writing—original draft preparation, A.S.H., M.A.Z. and R.H.D.S.; writing—review and editing, M.A.Z., R.H.D.S. and A.S.H.; visualization, A.S.H., M.A.Z. and R.H.D.S.; funding acquisition and project administration, R.H.D.S., A.S.H. and M.A.Z. All authors have read and agreed to the published version of the manuscript.

Funding: The first author wishes to acknowledge the financial support of the National Research Centre of Egypt (NRC). The second author wishes to acknowledge the support of RFBR Grant 19-01-00019.

Data Availability Statement: Data sharing is not applicable to this article.

Acknowledgments: The authors wish to thank the reviewers for their comments and criticism. All of their comments were taken into account in the revised version of the paper, resulting in a substantial improvement with respect to the original submission.

Conflicts of Interest: The authors declare no conflict of interest.

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